Intuitionistic modal proof theory (tutorial)

Sonia Marin

University of Birmingham, UK
s.marin@bham.ac.uk

Intuitionistic modal logic, despite more than seventy years of investigation [4], still partly escapes our comprehension.

Already answering what is the intuitionistic variant of normal modal logic K is not obvious. Lacking De Morgan duality, there are several variants of the normal k axiom that are classically but not intuitionistically equivalent. Five axioms have been considered as primitives in the literature. An intuitionistic variant of K can then be obtained from intuitionistic propositional logic IPL by

– adding the *necessitation rule*: $\Box A$ is a theorem if A is a theorem; and
– adding a subset of the following five axioms:

$$\begin{array}{llll} \mathsf{k}_1: & \Box(A \to B) \to (\Box A \to \Box B) & \mathsf{k}_3: & \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2: & \Box(A \to B) \to (\Diamond A \to \Diamond B) & \mathsf{k}_4: & (\Diamond A \to \Box B) \to \Box(A \to B) \\ & & \mathsf{k}_5: & \Diamond\bot \to \bot \end{array}$$

Structural proof theoretic accounts of intuitionistic modal logic have adopted either the paradigm of *labelled deduction* in the form of labelled natural deduction and sequent systems [6], or the one of *unlabelled deduction* in the form of sequent [2] or nested sequent systems [7, 1]. In this tutorial, we would like to give an overview of the current landscape of intuitionistic modal proof theory and illustrate how "old and new" approaches can complement each other.

We will review ordinary sequent calculi, which are adequate to treat logics based on a subset of k_1, k_2, k_3, and k_5, as well as *labelled* and *nested sequents*, which have been used to give deductive systems for the logics that cannot seem to be handled in ordinary sequent calculi, i.e., the ones that include k_4.

Both of these approaches (labelled and unlabelled) are still under active investigation. A framework for fragments of intuitionistic modal logics was recently designed, based on unlabelled sequents but related to a new intuitionistic version of *neighbourhood semantics* [3]. Another one proposes a refined labelled approach taking full advantage of the more standard *birelational semantics* [5]. As these allow for a fine-grained account of intuitionistic modal logic, we hope they will help shed some light on the intricacte world of intuitionistic modal logics.

References

1. Arisaka, R., Das, A., Straßburger, L.: On nested sequents for constructive modal logics. Logical Methods in Computer Science **11**(3) (2015)
2. Bierman, G.M., de Paiva, V.: Intuitionistic necessity revisited. Tech. rep., School of Computer Sciences-University of Birmingham (1996)

3. Dalmonte, T., Grellois, C., Olivetti, N.: Intuitionistic non-normal modal logics: A general framework. Journal of Philosophical Logic **49**(5), 833–882 (2020)
4. Fitch, F.B.: Intuitionistic modal logic with quantifiers **7**(2), 113–118 (1948)
5. Marin, S., Morales, M., Straßburger, L.: A fully labelled proof system for intuitionistic modal logics. Journal of Logic and Computation **31**(3), 998–1022 (2021)
6. Simpson, A.: The Proof Theory and Semantics of Intuitionistic Modal Logic. Ph.D. thesis, University of Edinburgh (1994)
7. Straßburger, L.: Cut elimination in nested sequents for intuitionistic modal logics. In: International Conference on Foundations of Software Science and Computational Structures. pp. 209–224. Springer (2013)