Cardinalities, Infinities and Choice Principles for Finitely Supported Sets

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Finitely supported sets and structures are related to the permutation models of Zermelo-Fraenkel set theory with atoms (ZFA) which were originally described to prove the independence of the axiom of choice from the other axioms of ZFA set theory. Since the existence of atoms (that are defined as entities having no internal structure) requires the modification of the Zermelo-Fraenkel (ZF) axiom of extensionality, finitely supported sets were alternatively described and studied in the ZF set theory by equipping ZF sets with actions of a group of one-to-one and onto transformations of some basic elements whose internal structure is ignored. These sets were used to model the renaming, variables binding and choosing fresh names in the theory of programming. More exactly, inductively defined finitely supported sets involving the name-abstraction together with Cartesian product and disjoint union can encode a formal syntax modulo renaming of bound variables. In this way, the standard theory of algebraic data types can be extended to include signatures involving binding operators. In particular, there exists an associated notion of structural recursion for defining syntax-manipulating functions and a notion of proof by structural induction. Certain generalizations of finitely supported sets are involved in the study of automata, programming languages or Turing machines over infinite alphabets; for this, a relaxed notion of finiteness called 'orbit finiteness' was defined; it means 'having a finite number of orbits (equivalence classes) under a certain group action'.

The notions of invariant set and finitely supported structure are introduced and described in previous articles of the authors; we particularly recommend our recent book [1] and its concise presentation [2]. Assume A is the set of basic elements (also called atoms by analogy with the ZFA approach). An invariant set $(X, \cdot)$ is a ZF set X equipped with a group action $\cdot$ of the group of all finitary permutations of A satisfying the requirement that every element of X is finitely supported under $\cdot$. An element $x \in X$ is finitely supported under $\cdot$ if there is a finite set $S_x \subset A$ such that any finitary permutation π of A fixing S_x pointwise has the property that $\pi \cdot x = x$. A finitely supported subset of the invariant set X (which is simply called finitely supported set) is a finitely supported element in the powerset of X equipped with the higher order permutation action $\star$ defined by $(\pi, Y) \mapsto \pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for π a finitary permutation of A and Y a subset of X. A finitely supported set Y is called uniformly supported if all the elements of Y are supported by the same finite set of atoms. The Cartesian product of two invariant sets $(X, \cdot)$ and $(Y, \diamond)$ is an invariant set with the action $(\pi, (x, y)) \mapsto (\pi \cdot x, \pi \diamond y)$. A relation (or, particularly, a function) between two finitely supported sets is finitely supported if it is finitely supported as a subset of the Cartesian product of those two finitely supported sets. Particularly, a function between two finitely supported sets $(X, \cdot)$ and $(Y, \diamond)$ is supported by a finite set S if and only if $f(\pi \cdot x) = \pi \diamond f(x)$, $\pi \cdot x \in X$ and $\pi \diamond f(x) \in Y$ for all $x \in X$ and all finitary permutations of atoms π that fix S pointwise. The set of all finitely supported functions from X to Y is denoted by Y^X_{fs}. A finitely supported function $f : A \to A$ is bijective if and only if it is a finitary permutation; thus, finitary permutations are simply called *permutations* in the framework of finitely supported structures. A finitely supported structure is a finitely supported set equipped with a finitely supported relation. The theory of finitely supported structures allows a discrete (finitary) representation of possibly infinite sets containing enough symmetries to be concisely handled. This theory allows us to treat as equivalent the elements in a structure that have a certain degree of similarity and to focus only on those elements that are really different forming the support of the structure.

The world of finitely supported structures contains both the family of non-atomic ZF structures which are proved to be trivially invariant (i.e. all their elements are empty supported since, intuitively, they are hierarchically constructed from $\emptyset$) and the family of atomic structures with finite, but possibly non-empty, supports. Our purpose is to check whether a ZF result remains valid when replacing 'non-atomic ZF structure' with 'atomic and finitely supported structure', and also to prove specific properties of finitely supported sets that have not a correspondent in the non-atomic framework. We emphasized in [1] that results from ZF might lose their validity when transferring them into an atomic framework such as ZFA. For example, 'Multiple choice principle implies the axiom of choice' is a valid theorem in ZF, but it does not hold in ZFA since multiple choice principle is valid in the Second Fraenkel Model of ZFA, while the axiom of choice is not valid in the related model. The meta-theoretical technique for transferring ZF results into the world of finitely supported sets and structures is based on a closure property for finite supports in a (higher-order) hierarchical construction, called 'S-finite support principle' claiming that *"for any finite set S of atoms, anything that can be defined in higher-order logic from structures supported by S, by using each time only constructions supported by S, is itself supported by S"*. The formal involvement of this meta-theoretical principle involves a step-by-step building of the support of a structure by employing, at every step, the previously constructed supports of the substructures of the related structure.

The results presented in this tutorial deal with three topics:

Results Regarding Choice Principles. The validity of choice principles in various models of Zermelo-Fraenkel set theory and of Zermelo-Fraenkel set theory with atoms (including the symmetric models and the permutation models) was investigated in the last century. Choice principles are proved to be independent from the axioms of Zermelo-Fraenkel set theory and of Zermelo-Fraenkel set theory with atoms, respectively. We were able to prove that the choice principles **AC** (axiom of choice), **HP** (Hausdorff maximal principle) **ZL** (Zorn lemma),

DC (principle of dependent choice), **CC** (principle of countable choice), **PCC** (principle of partial countable choice), **AC(fin)** (axiom of choice for finite sets), **Fin** (principle of Dedekind finiteness), **PIT** (prime ideal theorem), **UFT** (ultrafilter theorem), **OP** (total ordering principle), **KW** (Kinna-Wagner selection principle), **OEP** (order extension principle), **SIP** (principle of existence of right inverses for surjective mappings), **FPE** (finite powerset equipollence principle) and **GCH** (generalized continuum hypothesis) are not valid in the framework of finitely supported structures.

Results Regarding Cardinalities. The equipollence relation is an equivariant equivalence relation in the framework of finitely supported sets. For two finitely supported sets X and Y we say that they have the same cardinality, i.e. $|X| = |Y|$, if and only if there exists a finitely supported bijection $f : X \to Y$. Some arithmetic properties of cardinalities of finitely supported sets (regarding sums, products and exponents) are naturally translated from the non-atomic Zermelo-Fraenkel framework. However, we may have specific order properties. For example, on the family of cardinalities we can define the relations:

– $\leq$ by: $|X| \leq |Y|$ if and only if there is a finitely supported injective mapping $f : X \to Y$.
– $\leq^*$ by: $|X| \leq^* |Y|$ if and only if there is a finitely supported surjective mapping $f : Y \to X$.

We are able to prove that the relation $\leq$ is equivariant, reflexive, anti-symmetric and transitive, but it is not total, while the relation $\leq^*$ is equivariant, reflexive and transitive, but it is not anti-symmetric, nor total.

Results Regarding Infinities. We introduce and study various forms of infinity (of Tarski type, of Dedekind type, of Mostowski type, and so on) for finitely supported structures, and provide several relationship results between them. By presenting examples of atomic sets that satisfy a certain form of infinity, while they do not satisfy other forms of infinity, we were able to conclude that the definitions of infinity we introduce are pairwise non-equivalent.

Some formal results are summarized below (with X a finitely supported set):

1. X is called *classical infinite* if X does not correspond one-to-one and onto to a finite ordinal (it cannot be represented in the form $\{x_1, \ldots, x_n\}$).
2. X is *covering infinite* if there is a finitely supported directed family $\mathcal{F}$ of finitely supported sets with the property that X is contained in the union of the members of $\mathcal{F}$, but there does not exist $Z \in \mathcal{F}$ such that $X \subseteq Z$;
3. X is called *Tarski I infinite (TI i)* if there exists a finitely supported one-to-one mapping of X onto $X \times X$.
4. X is called *Tarski II infinite (TII i)* if there exists a finitely supported family of finitely supported subsets of X, totally ordered by inclusion, having no maximal element.
5. X is called *Tarski III infinite (TIII i)* if if there exists a finitely supported one-to-one mapping of X onto $X + X$.

6. X is called *Mostowski infinite (M i)* if there exists an infinite finitely supported totally ordered subset of X.
7. X is called *Dedekind infinite (D i)* if there exists a finitely supported one-to-one mapping of X onto a finitely supported proper subset of X.
8. X is called *ascending infinite (Asc i)* if there is a finitely supported increasing countable chain of finitely supported sets $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ with $X \subseteq \cup X_n$, but there does not exist $n \in \mathbb{N}$ such that $X \subseteq X_n$;
9. X is called *non-amorphous (N-am)* if X contains two disjoint, infinite, finitely supported subsets.

Some properties of Dedekind infinite sets are listed below.
Let X be a finitely supported set.

1. X is Dedekind infinite if and only if there exists a finitely supported one-to-one mapping $f : \mathbb{N} \to X$.
2. If X is classical infinite, then $\wp_{fs}(\wp_{fin}(X))$ is Dedekind infinite, where $\wp_{fin}(X)$ is the finite powerset of X.
3. If X does not contain an infinite uniformly supported subset, then $\wp_{fin}(X)$ does not contain an infinite uniformly supported subset, and so it is not Dedekind infinite.
4. If X does not contain an infinite uniformly supported subset, then the exponent set $X^{A^n}_{fs}$ does not contain an infinite uniformly supported subset, and so it is not Dedekind infinite, whenever $n \in \mathbb{N}$.
5. If $\wp_{fs}(X)$ is not Dedekind infinite, then each finitely supported surjective mapping $f : X \to X$ should be injective. The reverse implication is not valid because any finitely supported surjective mapping $f : \wp_{fin}(A) \to \wp_{fin}(A)$ is also injective, while $\wp_{fs}(\wp_{fin}(A))$ is Dedekind infinite.
6. If $\wp_{fin}(X)$ is Dedekind infinite, then X should contain two disjoint, infinite, uniformly supported subsets.
7. If $\wp_{fs}(X)$ is Dedekind infinite, then X contain two disjoint, infinite, finitely supported supported subsets. The reverse implication is not valid.
8. If X is a Dedekind infinite set, then there exists a finitely supported surjection $j : X \to \mathbb{N}$. The reverse implication is not valid.
9. If there exists a finitely supported bijection between X and $X + X$, then X contains an infinite uniformly supported subset. The reverse implication is not valid.

In Figure 1 we present some of the relationships between the definitions of infinite. The 'ultra thick arrows' symbolize *strict* implications (of from p implies q, but q does not imply p), while 'thin dashed arrows' symbolize implications for which we have not proved yet if they are strict or not (the validity of the reverse implications follows when assuming choice principles over non-atomic ZF sets). 'Thick arrows' match equivalences.

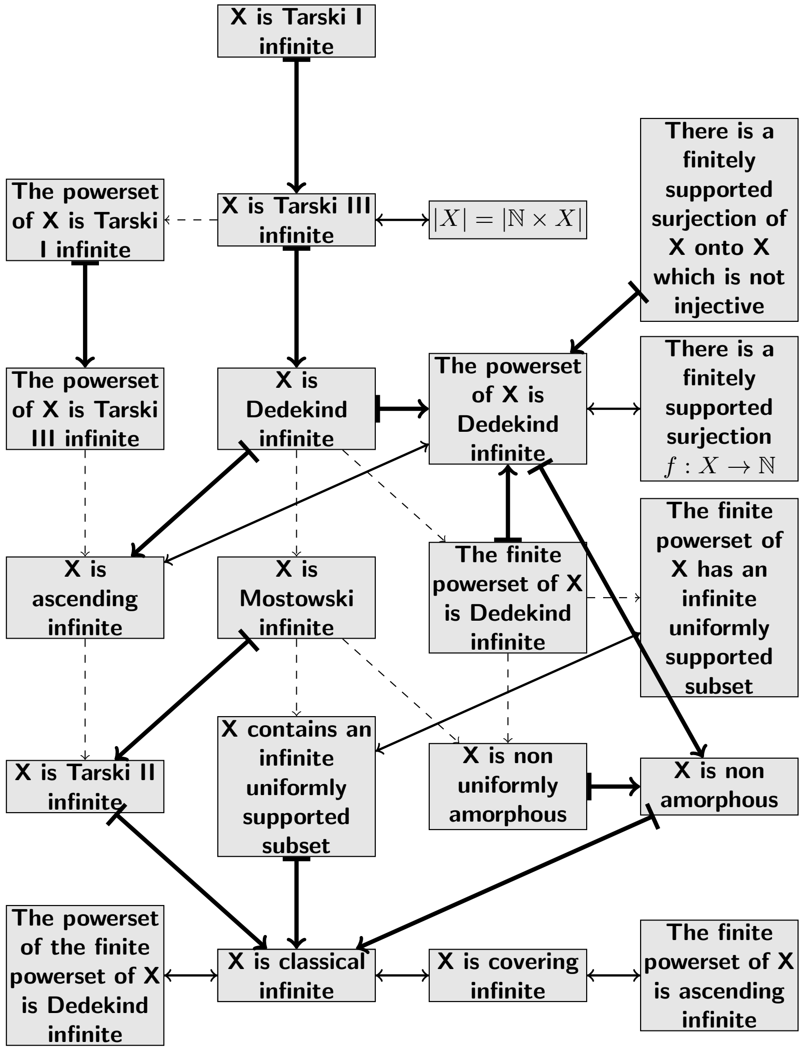

Fig. 1. Relationships between several forms of infinity.

Examples of some finitely supported sets satisfying various forms of infinity are presented in the table below:

Set	TI i	TIII i	D i	M i	Asc i	TII i	N-am
A	No	No	No	No	No	No	No
nA, $n > 1$	No	No	No	No	No	No	Yes
A^n, $n > 1$	No	No	No	No	No	No	Yes
$\wp_{fin}(A^n)$, $n > 0$	No	No	No	No	Yes	Yes	Yes
$T_{fin}(A^n)$	No	No	No	No	Yes	Yes	Yes
$\wp_{fs}(A^n)$	No	No	No	No	Yes	Yes	Yes
$\wp_{fin}(\wp_{fs}(A^n))$	No	No	No	No	Yes	Yes	Yes
$A^{A^n}_{fs}$	No	No	No	No	Yes	Yes	Yes
$T_{fin}(A)^{A^n}_{fs}$	No	No	No	No	Yes	Yes	Yes
$\wp_{fs}(A)^{A^n}_{fs}$	No	No	No	No	Yes	Yes	Yes
$A \cup \mathbb{N}$	No	No	Yes	Yes	Yes	Yes	Yes
$A \times \mathbb{N}$	No	Yes	Yes	Yes	Yes	Yes	Yes
$\wp_{fs}(A \cup \mathbb{N})$	No	Yes	Yes	Yes	Yes	Yes	Yes
$\wp_{fs}(\wp_{fs}(A))$	?	Yes	Yes	Yes	Yes	Yes	Yes
$A^{\mathbb{N}}_{fs}$ and $\mathbb{N}^{A}_{fs}$	Yes	Yes	Yes	Yes	Yes	Yes	Yes

References

1. Alexandru, A., Ciobanu, G.: *Foundations of Finitely Supported Structures: a set theoretical viewpoint*. Springer (2020).
2. Alexandru, A., Ciobanu, G.: Essentials of finitely supported structures *Bulletin of the European Association for Theoretical Computer Science* vol.133, 95-110 (2021)